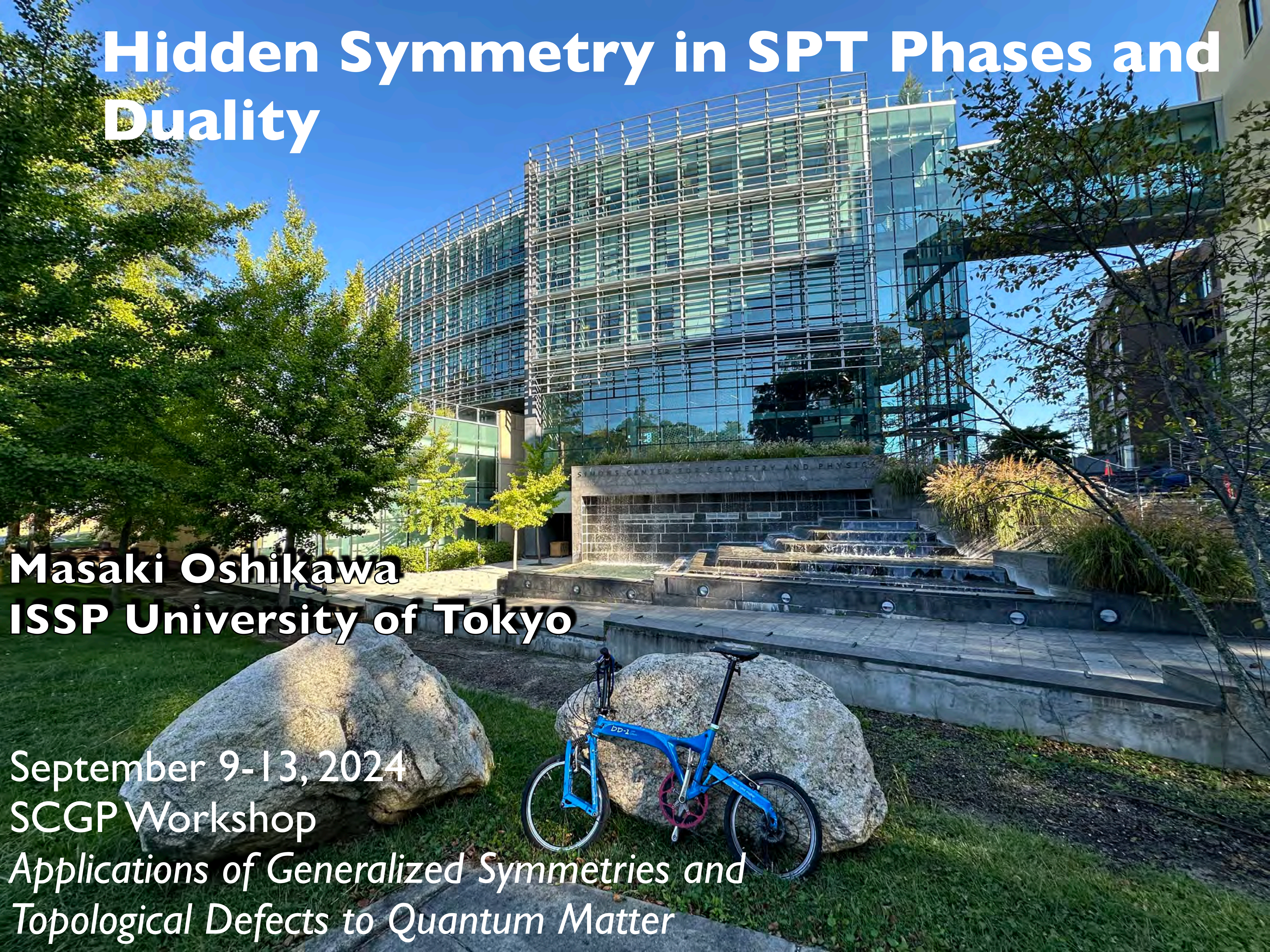
Hidden Symmetry in SPT Phases and Duality



Masaki Oshikawa
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September 9-13, 2024
SCGP Workshop
Applications of Generalized Symmetries and Topological Defects to Quantum Matter

Part I:
recap of what was known about "SPT" in the 20th century

Part II: recent progress

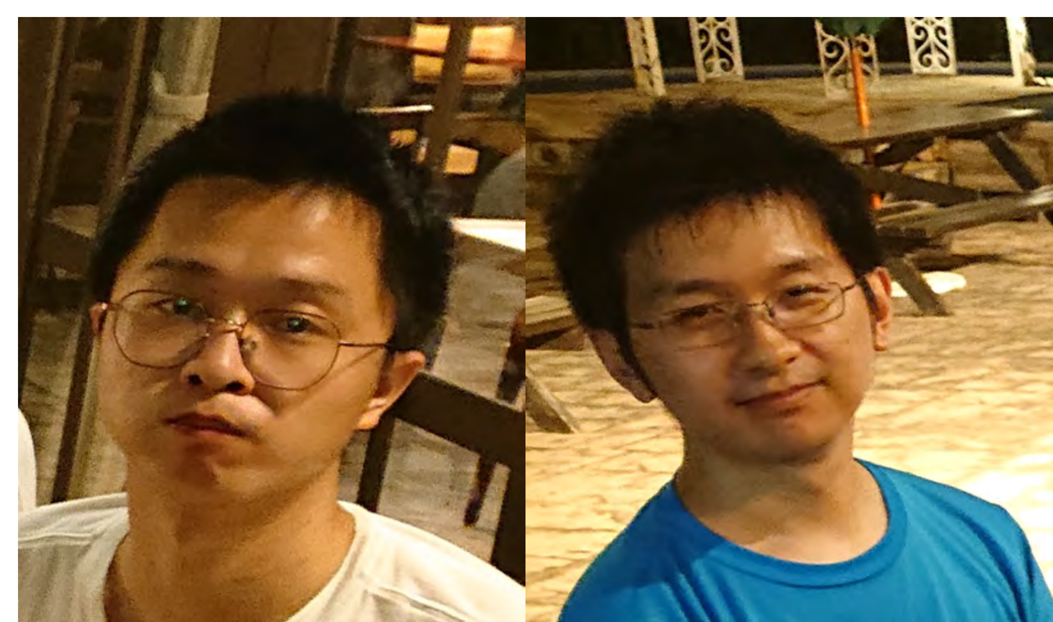
arXiv:2301.07899
(PRB 2024)

Non-Invertible Duality Transformation Between SPT and SSB Phases

arXiv:2307.04788

Intrinsically/Purely Gapless-SPT from Non-Invertible Duality Transformations

Linhao Li (ISSP→Ghent)
Yunqin Zheng
(ISSP/IPMU→Stony Brook)



Haldane gap

Heisenberg antiferromagnetic chain

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

S=1/2, 3/2, 5/2........

"massless" = gapless, power-law decay of spin correlations

S=1, 2, 3, ...

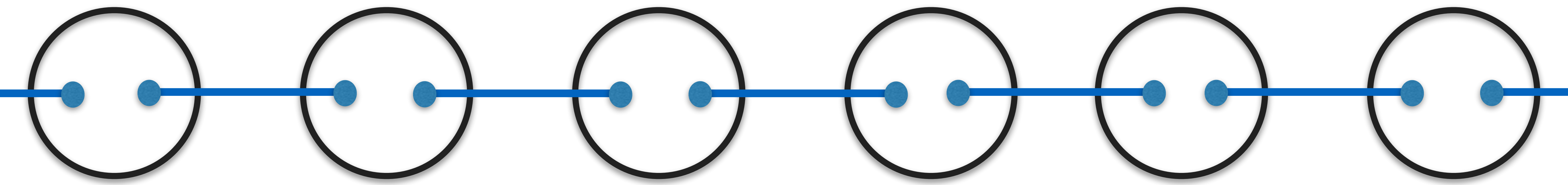
"massive" = non-zero gap, exponential decay of spin correlations

Haldane conjecture (1981)

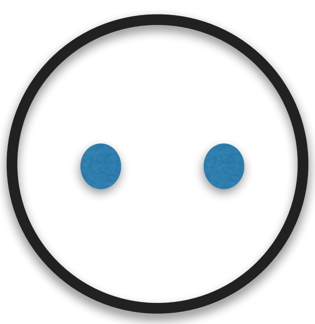
AKLT model/state

$$\mathcal{H} = J \sum_j \left[\vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} \left(\vec{S}_j \cdot \vec{S}_{j+1} \right)^2 \right]$$

Exact groundstate: (Affleck-Kennedy-Lieb-Tasaki 1987)



Singlet pair of two S=1/2's -"valence bonds"



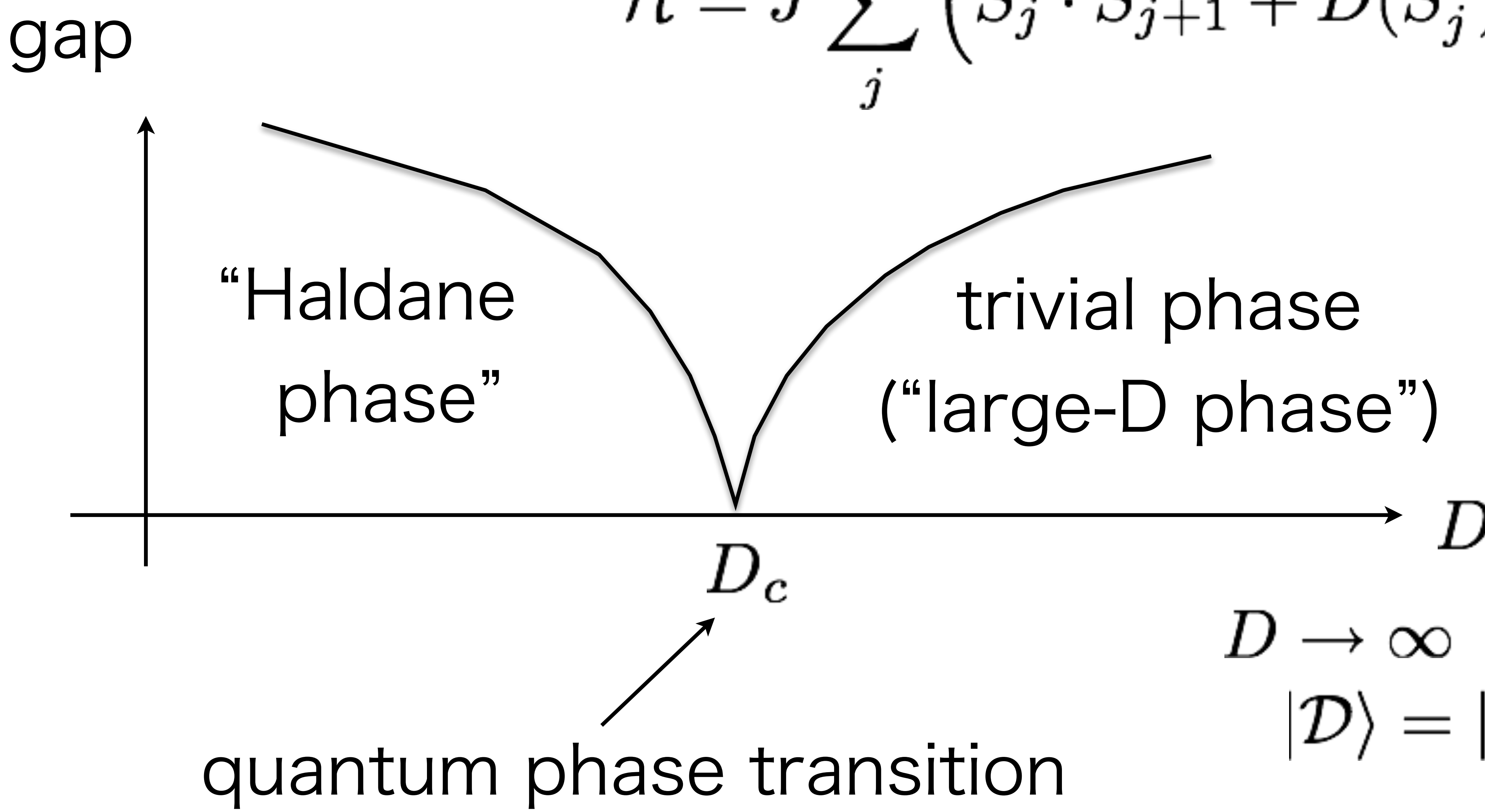
Symmetrization of two S=1/2's ⇒ S=1

✓non-zero gap, exponential decay of correlations

(supporting the Haldane conjecture)

Haldane Phase and QPT

$$\mathcal{H} = J \sum_j \left(\vec{S}_j \cdot \vec{S}_{j+1} + D(S_j^z)^2 \right).$$



$D \to \infty$

$|\mathcal{D}\rangle = |000000\ldots\rangle$

Why there is the transition?

Modern understanding: Haldane phase is a SPT!

Journal of the Physical Society of Japan
Vol. 65, No. 10, October, 1996, pp. 3317-3330

$$\mathcal{H}_0 = \sum_{\ell=1}^{N} \{1-(-1)^{\ell}\alpha\} \boldsymbol{S}_\ell \cdot \boldsymbol{S}_{\ell+1} + D\sum_{\ell=1}^{N} \left(S_\ell^z\right)^2,$$

Ground-State Phase Diagram and Magnetization Curves of the Spin-1 Antiferromagnetic Heisenberg Chain with Bond Alternation and Uniaxial Single-Ion-Type Anisotropy

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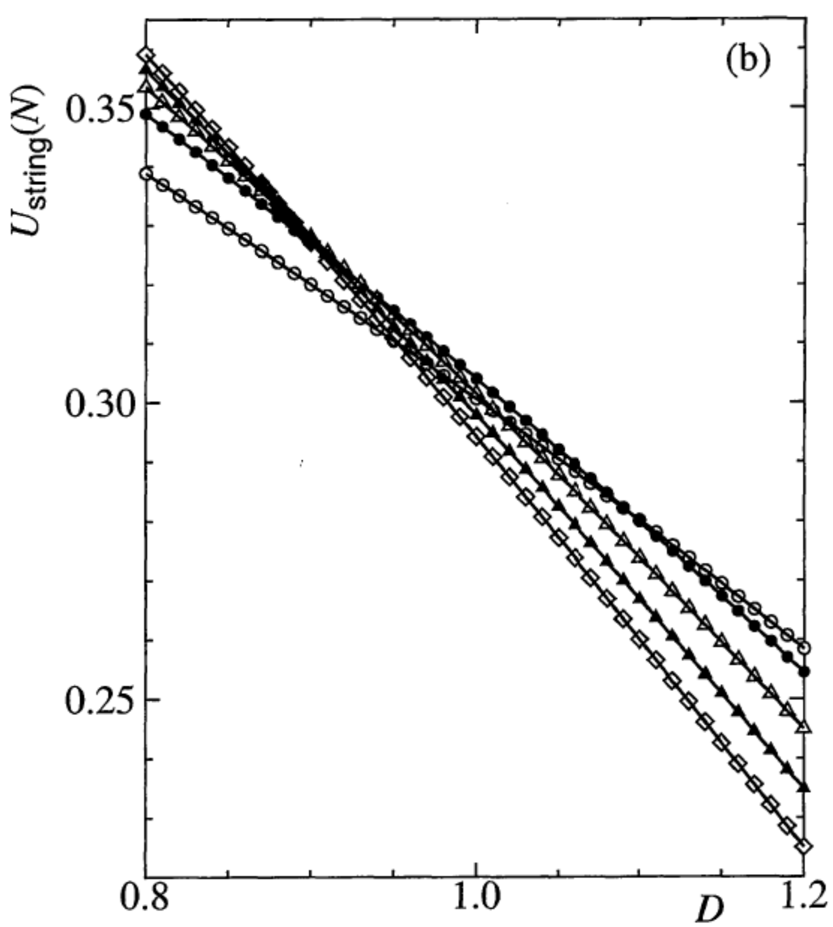
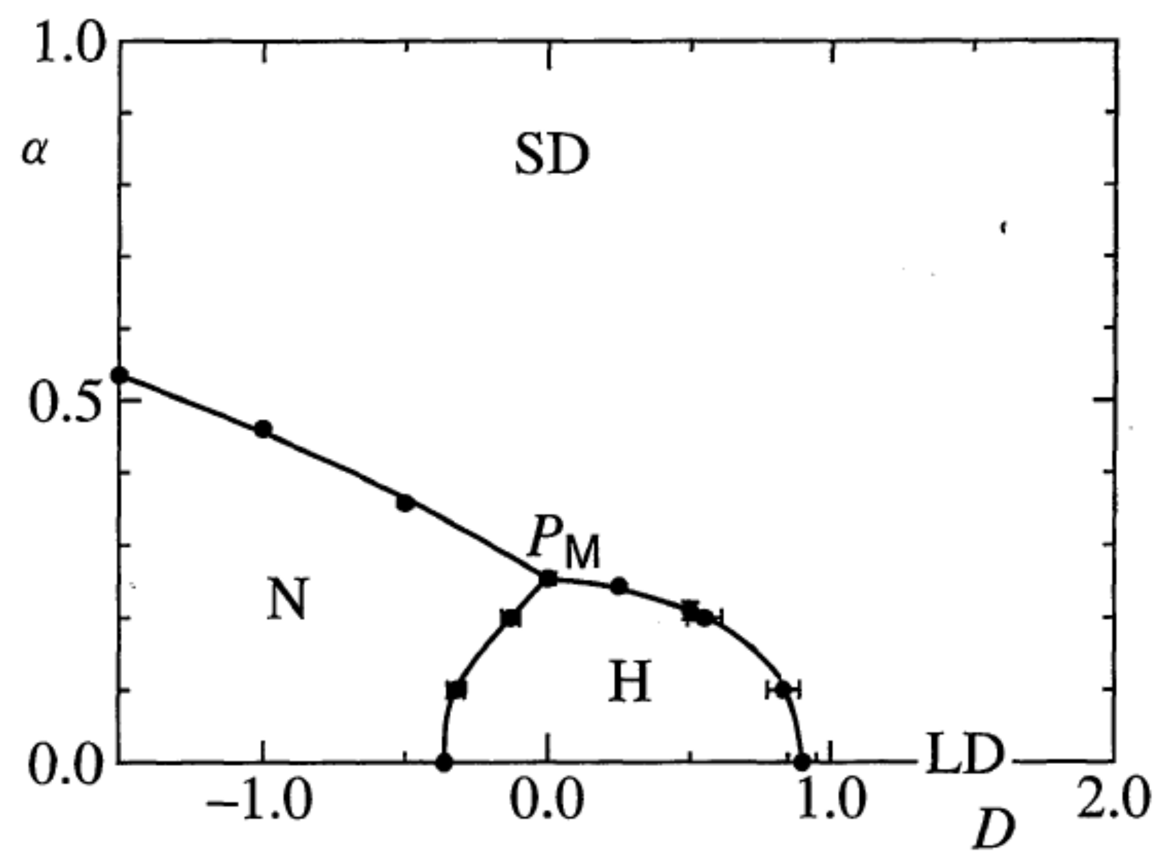
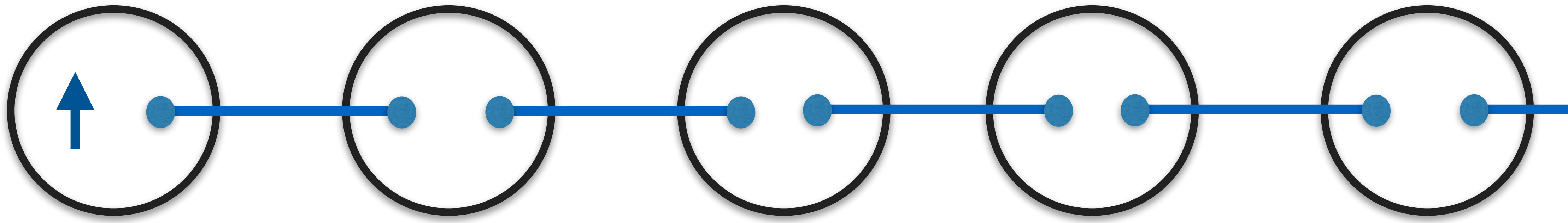
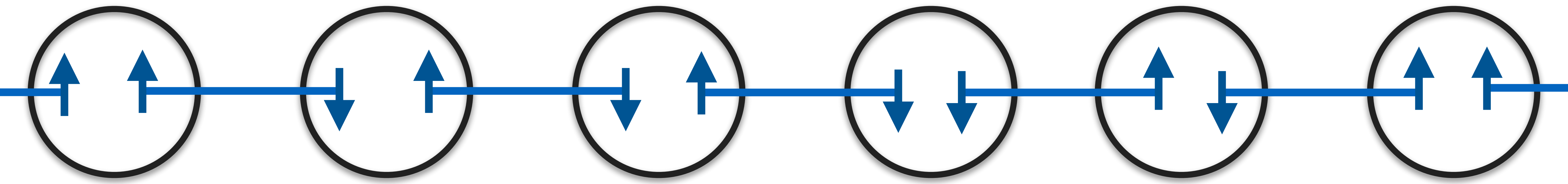




Nontrivial Features of Haldane Phase



Open boundary condition :"edge state" of $S=1/2$ [Kennedy 1990]



$S^z = +1$ 0 0 -1 0 +1

non-local "string" order

[den Nijs-Rommelse 1989]

$$\mathcal{O}^z_{str} = \lim_{|j-k|\to\infty} \langle S^z_j \exp\left(i\pi \sum_{l=j}^{k-1} S^z_l\right) S^z_k \rangle$$

J. Phys.: Condens. Matter **2** (1990) 5737–5745. Printed in the UK

Exact diagonalisations of open spin-1 chains

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$$H = \sum_{i=1}^{L-1} [S_i \cdot S_{i+1} - \beta(S_i \cdot S_{i+1})^2]$$

Received 3 January 1990

Abstract. We numerically compute the two lowest eigenvalues of finite length spin-1 chains with the Hamiltonian $H = \Sigma_i[S_i \cdot S_{i+1} - \beta(S_i \cdot S_{i+1})^2]$ and open boundary conditions. For a range of β, including the value 0, we find that the difference of the two eigenvalues decays exponentially with the length of the chain. This exponential decay provides further evidence that these spin chains are in a massive phase as first predicted by Haldane. The correlation length ξ of the chain can be estimated using this exponential decay. We find estimates of ξ for the Heisenberg chain ($\beta = 0$) that range from 6.7 to 7.8 depending on I
infinite length.

Table 1. The difference of the two lowest eigenvalues of the chain with open b
conditions and L sites for several $\beta > -\frac{1}{3}$.

L	$\beta = -0.30$	$\beta = -0.20$	$\beta = -0.10$	$\beta = 0.00$	$\beta = 0.40$
4	0.039379	0.171777	0.329668	0.509170	1.331250
5	0.023224	0.147423	0.328731	0.546645	1.603914
6	0.011608	0.082462	0.184574	0.307786	0.927374
7	0.006364	0.066161	0.178889	0.330956	1.153717
8	0.003310	0.040809	0.110734	0.201879	0.696740
9	0.001774	0.031059	0.102979	0.212703	0.877509
10	0.000935	0.020254	0.068391	0.138331	0.548516
11	0.000497	0.014880	0.061122	0.141772	0.693475
12	0.000263	0.010015	0.042723	0.097142	0.445742
13	0.000140	0.007200	0.036921	0.096709	0.563294
14	—	0.004933	—	0.069165	—

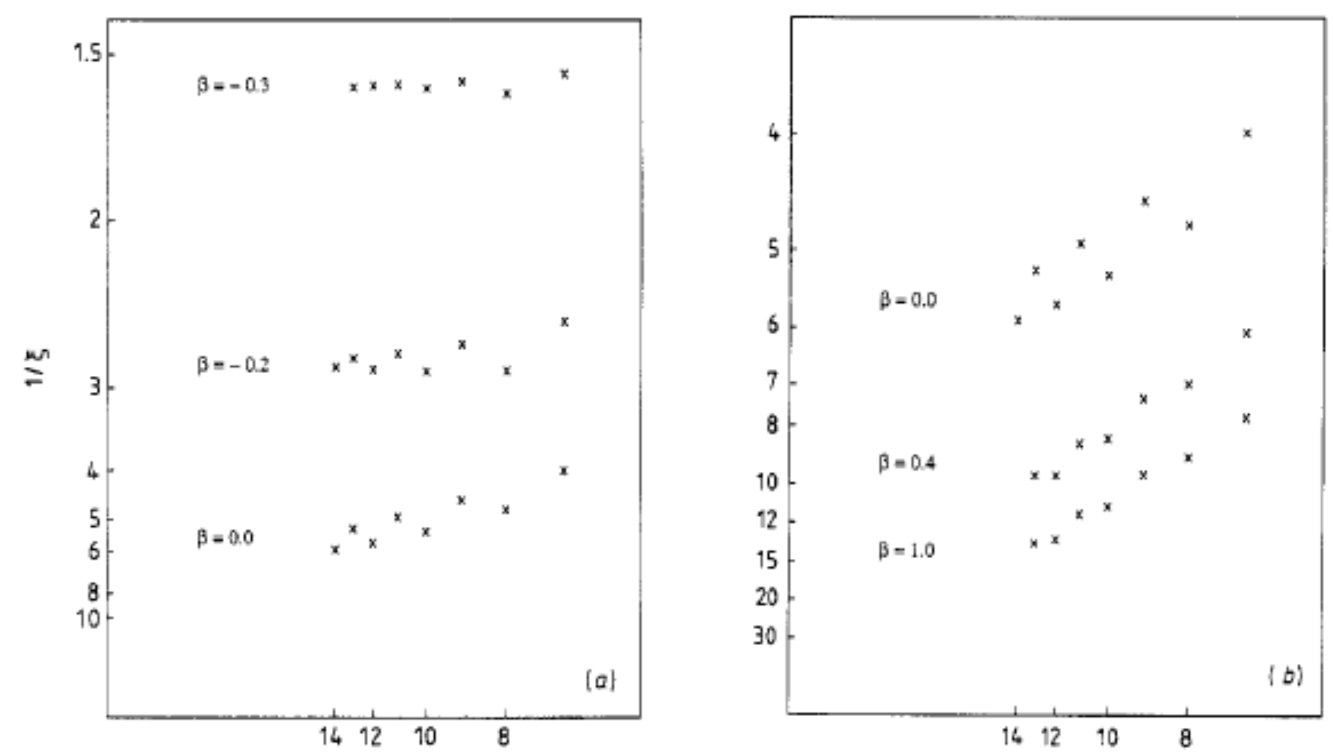

Figure 1. Plot of $\frac{1}{2}\ln(s_{L-2}/s_L)$ as a function of $1/L$, where s_L is the difference between the two lowest eigenvalues. The intersection of these curves with the vertical axis gives the inverse correlation length, so several correlation lengths are marked on the vertical axis. The horizontal axis is $1/L$, but the labels are values of L.

PHYSICAL REVIEW B VOLUME 40, NUMBER 7 1 SEPTEMBER 1989

Preroughening transitions in crystal surfaces and valence-bond phases in quantum spin chains

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(Received 10 April 1989)

We show that disordered flat phases in crystal surfaces are equivalent to valence-bond-type phases in integer and half-integer spin quantum chains. In the quantum spin representation the disordered flat phase represents a fluid-type phase with long-range antiferromagnetic spin order. This order is stabilized dynamically by the hopping of the particles and short-range spin-exchange interactions. The mass of Néel solitons is finite. Numerical finite-size-scaling results confirm this. We identify the order parameter of the valence-bond phase. The Haldane conjecture suggests a fundamental difference between half-integer and integer antiferromagnetic Heisenberg spin chains. We find that disordered flat phases are realized in both cases, have exactly the same type of long-range antiferromagnetic spin order, and are stabilized by exactly the same mechanism. They differ only in the mathematical formulation of broken symmetry in the spin representation. We suggest experimental methods of observing disordered flat phases in crystal surfaces.

It is impossible to define local order parameters that distinguish these two phases. The local order parameters of Sec. II E become nonlocal string operators in the spin-1 formulation (where the surface configuration is characterized by the steps). Recall the Ising-type order parame-

rameter ψ, Eq. (2.8), which van-
in the DOF and BCSOS flat

$$\left(i\pi \sum_{m=1}^{n} S^z_M \right) S^z_n \left| 0 \right\rangle , \qquad (4.4)$$

and its square is the limiting value of the correlation function, Eq. (2.7),

$$G_s(n) = \left\langle 0 \left| S^z_{n_0} \exp\left(i\pi \sum_{m=n_0}^{n+n_0} S^z_m \right) S^z_{n+n_0} \right| 0 \right\rangle . \qquad (4.5)$$

Physica Scripta. Vol. T27, 156–159, 1989.

Hidden Topological Order in Integer Quantum Spin Chains

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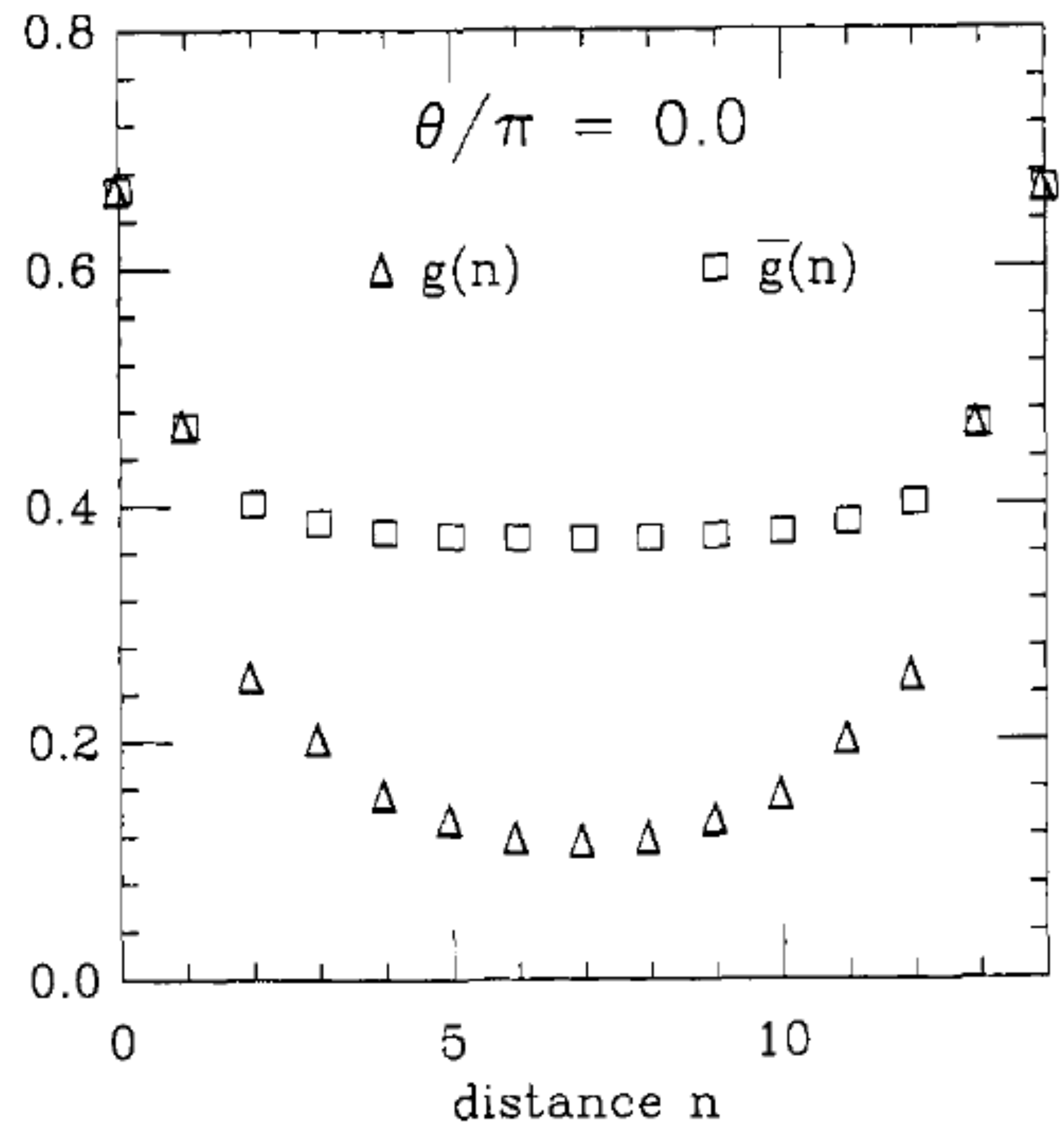
Received June 8, 1988, accepted July 25, 1988

3. Conclusions

We have investigated analogies between integer quantum spin-chains and the fractional quantum Hall effect. Both systems appear to have disordered liquid ground states but because of subtle topological effects, they both have an excitation gap. This topological order is not visible in the ordinary two-point correlation function, but can be detected by defining a special singular-gauge correlation function.



PHYSICAL REVIEW B VOLUME 45, NUMBER 1 1 JANUARY 1992-I

Hidden $Z_2 \times Z_2$ symmetry breaking in Haldane-gap antiferromagnets

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(Received 29 July 1991)

We show that the Haldane phase of the $S=1$ antiferromagnetic chain is closely related to the breaking of a hidden $Z_2 \times Z_2$ symmetry. When the chain is in the Haldane phase, this $Z_2 \times Z_2$ symmetry is fully broken, but when the chain is in a massive phase other than the Haldane phase, e.g., the Ising phase or the dimerized phase, this symmetry is broken only partially or not at all. The hidden $Z_2 \times Z_2$ symmetry is revealed by introducing a nonlocal unitary transformation of the chain. This unitary transformation also leads to a simple variational calculation which qualitatively reproduces the phase diagram of the $S=1$ chain.

edge states
string order : consequences of hidden Z2xZ2 symmetry

Haldane(SPT) vs trivial phases ↔ SSB vs trivial phases

How it works

$$U_{\mathrm{KT}} = \prod_{j<k} \exp\left(i\pi S_j^z S_k^x\right)$$

[simplified expression in M.O. 1992]

$$U_{\mathrm{KT}} S_j^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l<j} S_l^z\right) S_j^z$$

$$U_{\mathrm{KT}} S_j^x U_{\mathrm{KT}}^\dagger = S_j^x \exp\left(i\pi \sum_{j<l} S_l^x\right)$$

Kennedy-Tasaki transformation is a well-defined **unitary** for a finite chain with the open boundary condition which we assume for the moment (will come back later)

KT transformation of H

$$\mathcal{H} = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D\sum_j \left(S_j^z\right)^2$$

$$\begin{aligned}\tilde{\mathcal{H}} &= U_{\mathrm{KT}}\mathcal{H}U_{\mathrm{KT}}^\dagger \\ &= J\sum_j \left(S_j^x e^{i\pi S_{j+1}^x} S_{j+1}^x + S_j^y e^{i\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y + S_j^z e^{i\pi S_j^z} S_{j+1}^z\right) + D\sum_j \left(S_j^z\right)^2\end{aligned}$$

lacks the global SU(2) spin rotation symmetry, but still has the discrete global symmetry (π-rotation about x, y, & z axes)

dihedral group D2 = Z_2 x Z_2

Consequence of Dual SSB (I)

Suppose that the full global D_2 symmetry of the dual system is spontaneously broken

dual system: $$\langle S_j^z S_k^z \rangle \to \text{const.} \neq 0 \quad (k - j \to \infty)$$

⇕

original system:

$$\langle S_j^z \exp\left(i\pi \sum_{j \le l < k} S_l^z \right) S_k^z \rangle \to \text{const.} \neq 0 \quad (k - j \to \infty)$$

long-range "string order"!!

Consequences of Dual SSB (II)

Full global D2 symmetry of the dual system is spontaneously broken

↓

Dual system has 4-fold (quasi-)degenerate ground states

↓

Original system also has 4-fold (quasi-)degenerate ground states (only) for the open boundary condition

Edge state!

Modern View of the KT Duality

THE question lacking in 1990s:

When does the hidden $Z_2 \times Z_2$ symmetry breaking argument work?

Pollmann-Berg-Turner-MO, arXiv:0909.4059

Hidden $Z_2 x Z_2$ symmetry breaking is useful
iff the dual Hamiltonian is local (short-range int.)
⇔ the original Hamiltonian has global $Z_2 \times Z_2$ symmetry

If the Hamiltonian has the global $Z_2 \times Z_2$ symmetry,
the phase with the SSB of the hidden $Z_2 \times Z_2$ symmetry
is well-defined and separated from the trivial phase by
a quantum phase transition = **$Z_2 \times Z_2$ protected SPT!!**

Digression

Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin

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Received 9 December 1991, in final form 4 June 1992

$$V = V^{-1} = \prod_{j<k} \exp(i\pi S_j^z S_k^x)$$

explicitly for several variants of the VBS-type states. In the standard VBS state, the hidden $Z_2 \times Z_2$ symmetry breaks down when S is odd but remains unbroken when S is even. Our results for partially dimerized VBS states suggest that the hidden $Z_2 \times Z_2$

"Hidden $Z_2 \times Z_2$ symmetry breaking" in Haldane gap phase
for S=1 ⇒ nontrivial SPT [Gu-Wen 2009]

only if S is odd [Pollmann et al 2009]

Kennedy-Tasaki Duality in 21st Century

Many of the nontrivial features of the "Haldane gap phase" were recognized and unified as a consequence of "hidden symmetry breaking" in 1990s although the concept of SPT was (just) missing

Great progress in understanding SPT phases since the discovery/proposal in 2009

Revisit the Kennedy-Tasaki duality with the modern understanding
- reformulation of the Kennedy-Tasaki duality
- applications, especially to construction of gapless SPTs

KT transformation on a ring?

$$U_{\mathrm{KT}} S_j^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l<j} S_l^z\right) S_j^z$$

$$U_{\mathrm{KT}} S_{L+1}^z U_{\mathrm{KT}}^\dagger = \boxed{\exp\left(i\pi \sum_{l=1}^{L} S_l^z\right)} S_{L+1}^z$$

generator of global Z2xZ2 symmetry!

$$R^x = \exp\left(i\pi \sum_{l=1}^{L} S_l^z\right) = (-1)^{u_x} \qquad S_{L+1}^z = (-1)^{t_z} S_1^z$$

Dual spins obey:

$$u'_z = u_z \mod 2$$
$$t'_z = t_z + u_x \mod 2,$$

Two Interpretations

1) boundary conditions for the original & dual spins are given
   → only the "right" symmetry sector survives
   → **KT transformation is non-invertible**/non-unitary

2) boundary condition (periodic/twisted) is an auxiliary degree of freedom
   → separate Hilbert spaces for periodic/twisted b.c.
   → KT transformation is unitary on the extended Hilbert space

cf.) similar phenomena in Kramers-Wannier duality
only the "right" symmetry sector survives on a ring

Kramers-Wannier duality can be defined as
a unitary transformation on an open chain

Field-Theory Formulation

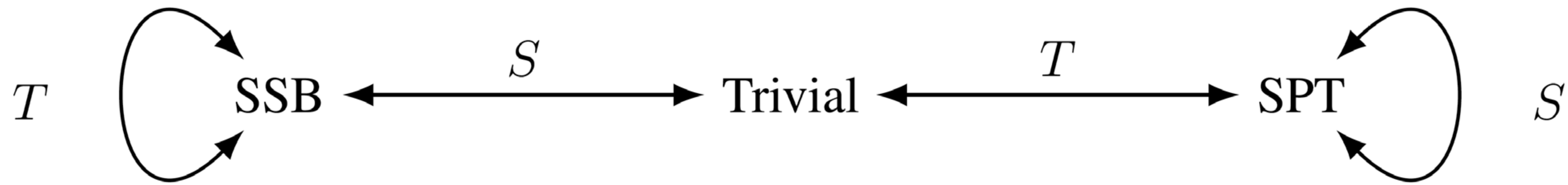
topological manipulations

S: gauging Z2xZ2

$$S : Z_{S_{12}\mathcal{X}}[A_1, A_2] := \frac{1}{|H^0(X_2, \mathbb{Z}_2)|^2} \sum_{a_1, a_2 \in H^1(X_2, \mathbb{Z}_2)} Z_{\mathcal{X}}[a_1, a_2](-1)^{\int_{X_2} a_1 A_2 + a_2 A_1}$$

T: stacking a Z2xZ2 SPT

$$T : Z_{T_{12}\mathcal{X}}[A_1, A_2] := Z_{\mathcal{X}}[A_1, A_2](-1)^{\int_{X_2} A_1 A_2}.$$



Kennedy-Tasaki = *STS*

how to implement this on lattice?

KT transformation for S=1/2

consider a system of two species of S=1/2: σ and τ

Z2xZ2 symmetry generated by $$U_\sigma = \prod_{i=1}^{L} \sigma_i^x, \qquad U_\tau = \prod_{i=1}^{L} \tau_{i-\frac{1}{2}}^x .$$

S: gauging by Z2xZ2 ⇔ Kramers-Wannier for σ, τ

$$\mathcal{N}\,|\{s_i^\sigma, s_{i-\frac{1}{2}}^\tau\}\rangle = \frac{1}{2^L} \sum_{\{\widehat{s}_{j-\frac{1}{2}}^\sigma, \widehat{s}_j^\tau\}} (-1)^{\sum_{j=1}^{L} s_j^\sigma(\widehat{s}_{j-\frac{1}{2}}^\sigma + \widehat{s}_{j+\frac{1}{2}}^\sigma) + t_\sigma \widehat{s}_{\frac{1}{2}}^\sigma + \widehat{s}_j^\tau(s_{j-\frac{1}{2}}^\tau + s_{j+\frac{1}{2}}^\tau) + \widehat{t}_\tau s_{\frac{1}{2}}^\tau}\,|\{\widehat{s}_{j-\frac{1}{2}}^\sigma, \widehat{s}_j^\tau\}\rangle$$

T: stacking with Z2xZ2 ⇔ "Domain wall decoration"

$$U_{\rm DW}\,|\{\widehat{s}_{i-\frac{1}{2}}^\sigma, \widehat{s}_i^\tau\}\rangle = (-1)^{\sum_{j=1}^{L} \widehat{s}_j^\tau(\widehat{s}_{j-\frac{1}{2}}^\sigma + \widehat{s}_{j+\frac{1}{2}}^\sigma) + \widehat{t}_\tau \widehat{s}_{\frac{1}{2}}^\sigma}\,|\{\widehat{s}_{i-\frac{1}{2}}^\sigma, \widehat{s}_i^\tau\}\rangle .$$

$$\mathcal{N}_{\rm KT} = \mathcal{N}\, U_{\rm DW}\, \mathcal{N} .$$

Symmetry/Twist Sectors

Symmetry sectors for σ, τ

$u_{\sigma,\tau}$ = 0, 1 (even/odd under spin flip)

Twist sectors for σ, τ

$t_{\sigma,\tau}$ = 0, 1 (periodic/antiperiodic boundary condition on ring)

dual spin original spin

$$(u'_\sigma, u'_\tau, t'_\sigma, t'_\tau) = (u_\sigma, u_\tau, u_\tau + t_\sigma, u_\sigma + t_\tau).$$

Similar to the original KT for S=1 $t'_z = t_z + u_x \mod 2,$

(in fact we have shown the equivalence between the KTs)

Construction of SPT

Two decoupled Ising chains in the ordered phase

$$H_{\mathrm{SSB}} = -\sum_{i=1}^{L}\left(\sigma^z_{i-1}\sigma^z_i + \tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}}\right)$$

Z2xZ2 fully broken spontaneously

Kennedy-Tasaki duality mapping

$\mathcal{N}_{\mathrm{KT}}$

$$H_{\mathrm{SPT}} = -\sum_{j=1}^{L}\left(\sigma^z_{j-1}\tau^x_{j-\frac{1}{2}}\sigma^z_j + \tau^z_{j-\frac{1}{2}}\sigma^x_j\tau^z_{j+\frac{1}{2}}\right),$$

1D “cluster model”: Z2xZ2 SPT

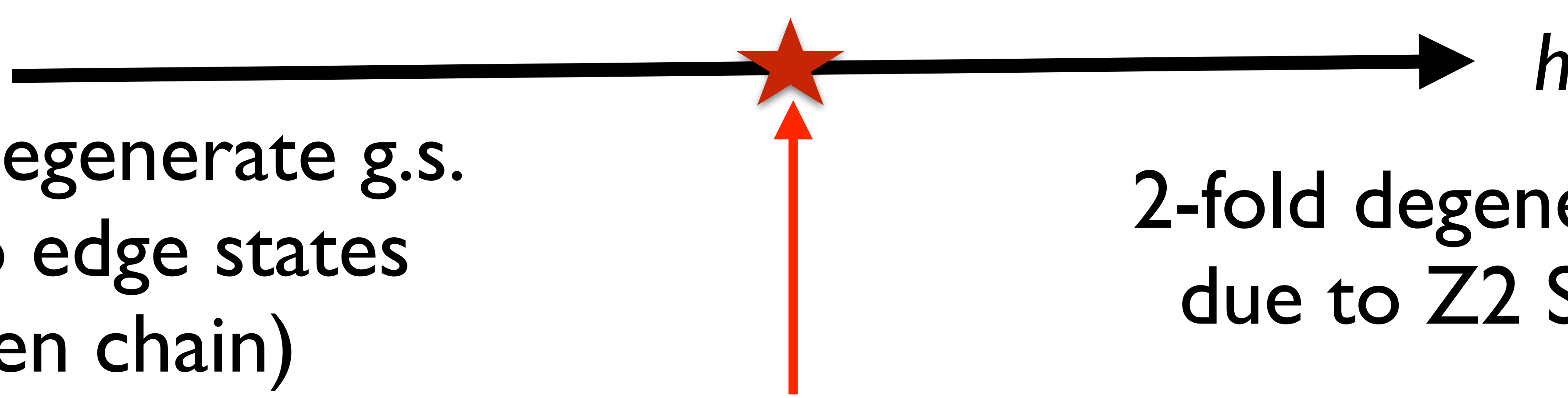
SPT-SSB Phase Transition

$$H = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \tau^x_{i-\frac{1}{2}} \sigma^z_i + h\sigma^x_i \right)$$

Z2xZ2 SPT

critical point

Z2 SSB (of τ)



4-fold degenerate g.s. due to edge states (on open chain)

2-fold degenerate g.s. due to Z2 SSB

2-fold (exponentially) degenerate g.s. due to edge states — "gapless SPT"

[Scaffidi, Parker, Vasseur 2017]

Duality Viewpoint

$$H = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \tau^x_{i-\frac{1}{2}} \sigma^z_i + h\sigma^x_i \right)$$

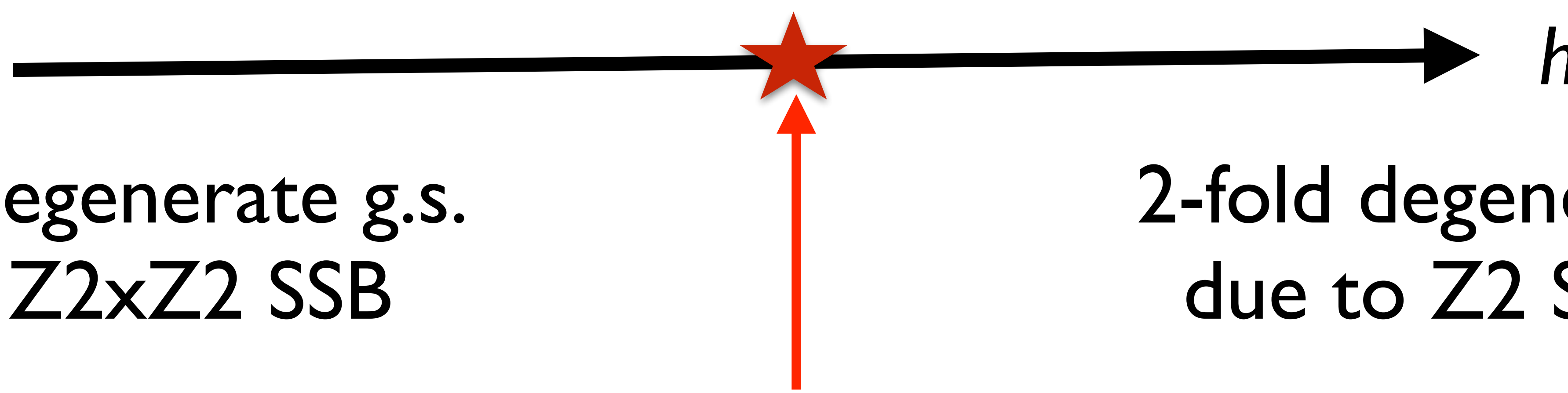
$$\Updownarrow \quad \mathcal{N}_{\mathrm{KT}}$$

$$H_{\mathrm{dual}} = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \sigma^z_i + h\sigma^x_i \right)$$

Z2xZ2 SSB

critical point

Z2 SSB (of τ)



4-fold degenerate g.s.
due to Z2xZ2 SSB

2-fold degenerate g.s.
due to Z2 SSB

2-fold (exponentially) degenerate g.s.
remaining due to “spectator” SSB of τ

Intrinsically Gapless SPT

Verresen, Thorngren, Jones, Pollmann, 2019
Thorngren, Vishwanath, Verresen 2020
Li-MO-Zheng 2022, Wen-Potter 2022 etc.

"topological" features of the gapless SPT phase has no counterpart in a gapped SPT

Entire global symmetry G: non-anomalous
subgroup G_{low} of G acts on low-energy sector anomalously
(cancelled by anomaly in the gapped sector)

Intrinsically Gapless SPT

$$H_{\mathrm{SSB+XX}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\sigma^z_i\right).$$

$$\Updownarrow \quad \mathcal{N}_{\mathrm{KT}}$$

$$H_{\mathrm{igSPT}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\tau^x_{i-\frac{1}{2}}\sigma^z_i\right)$$

"intrinsically gapless SPT" protected by Z_4 symmetry generated by $U_\sigma V_\tau$

$$U_\sigma = \prod_j \sigma^x_j \qquad V_\tau = \prod_{i=1}^{L} e^{\frac{i\pi}{4}(1-\tau^x_{i-\frac{1}{2}})}$$

igSPT + Z_4 symmetric perturbation

$$H_{\text{igSPT+pert}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\tau^x_{i-\frac{1}{2}}\sigma^z_i + h\sigma^x_i + h\tau^x_{i-\frac{1}{2}}\right)$$

h respects the Z4 symmetry

the system is trivial in the limit $h \to \infty$

is the igSPT phase stable against a small h? phase diagram?

$\mathcal{N}_{\mathrm{KT}}$

$$H_{\text{XX+pert}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\right)$$

XY chain in a field

$$H_{\text{SSB+pert}} = -\sum_{i=1}^{L}\left(\sigma^z_{i-1}\sigma^z_i + h\sigma^x_i\right).$$

Transverse Ising chain

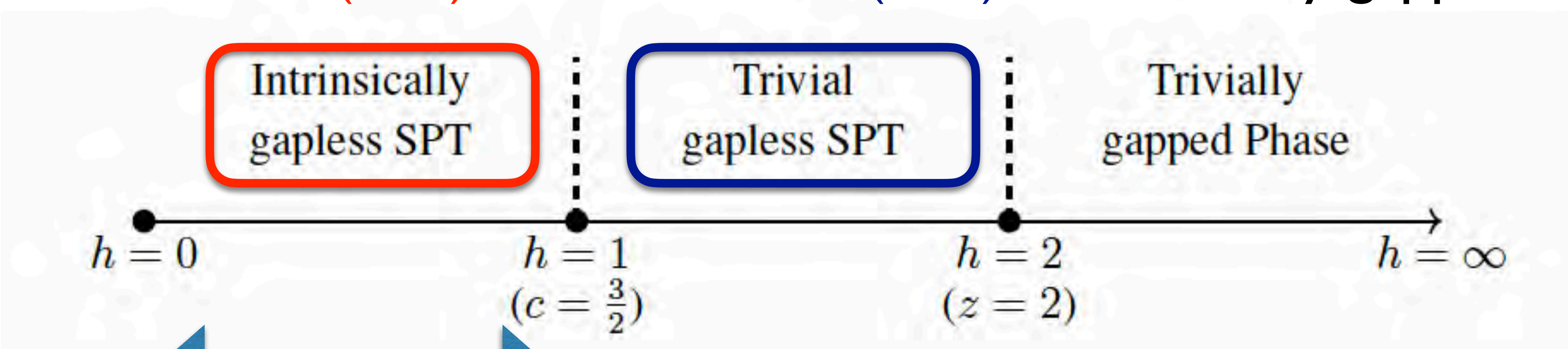
Both exactly solvable!

Phase Diagram

Ising SSB
XY critical (TLL)

Ising trivial
XY critical (TLL)

Ising trivial
XY trivially gapped



$$\langle \sigma_i^z \left(\prod_{k=i}^{j-1} \tau^x_{k+1/2} \right) \sigma_j^z \rangle \sim O(1)$$

$$\langle \tau^z_{i-1/2} \left(\prod_{k=i}^{j-1} \sigma^x_k \right) \tau^z_{j-1/2} \rangle \sim \frac{1}{|i-j|^{2\Delta}}$$

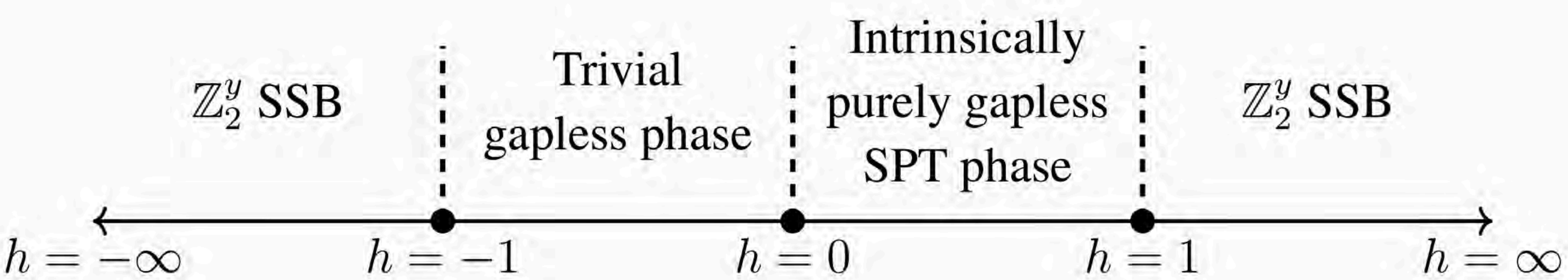
Getting Rid of Gapped Sector?

Replace the gapped SSB in the dual system with a gapless system

$$H^h_{\mathrm{XXZ+XXZ}} = -\sum_{i=1}^{L}\left(\sigma^z_i\sigma^z_{i+1} + \sigma^y_i\sigma^y_{i+1} + h\sigma^x_i\sigma^x_{i+1} + \tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\tau^x_{i+\frac{1}{2}}\right).$$

$$\Big\Updownarrow \; \mathcal{N}_{\mathrm{KT}}$$

$$H^h_{\mathrm{ipgSPTpert}} = -\sum_{i=1}^{L}\left(\sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1} + \sigma^y_i\tau^x_{i+\frac{1}{2}}\sigma^y_{i+1} + h\sigma^x_i\sigma^x_{i+1} + \tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\tau^x_{i+\frac{1}{2}}\right).$$



(ipgSPT characterized by symmetry charges in twisted sectors)

Summary

$\mathcal{N}_{\mathrm{KT}}$

trivial + trivial	⟷	trivial
Z2 SSB + trivial	⟷	Z2 SSB + trivial
Z2 SSB + Z2 Ising CFT	⟷	Z2xZ2 gapless SPT
Z2 SSB + Z2 SSB	⟷	Z2xZ2 SPT
Z2 SSB + Z4 free boson CFT	⟷	Z4 intrinsically gapless SPT
Z2 free boson CFT + Z2 free boson CFT	⟷	Z2xZ2 purely gapless SPT
Z2 free boson CFT + Z4 free boson CFT	⟷	Z4 intrinsically purely gapless SPT

Recent Developments

arXiv:2311.90050 systematic classification of gSPT with dualities

Classification of 1+1D gapless symmetry protected phases via topological holography

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(Dated: November 2, 2023)

arXiv:1803.02369, arXiv:2402.09520
duality for subsystem symmetries

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Subsystem symmetry protected topological order

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Kennedy-Tasaki transformation and non-invertible symmetry in lattice models beyond one dimension

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(Dated: February 16, 2024)

arXiv:2403.00905 duality for fusion category symmetries

Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries

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